Singular Perturbations for a Mayer Variational Problem

HENRY J. KELLEY*
Analytical Mechanics Associates Inc., Jericho, N.Y.

Introduction

ASYMPTOTIC expansions are of interest for approximate solution of optimal trajectory and control problems; however, these are two-point boundary value problems for the Euler equations, whereas the most highly developed theory for nonlinear differential equations[1] is for initial value problems. The present Note reports on an exploratory look at the simplest variational problem that one might expect to solve approximately in terms of a reduced-order solution plus boundary layers at each end. The primary objective is to adapt the available initial value machinery to this problem. Familiarity with the material in Chap. 10 of Ref. 1 will be assumed in the interest of brevity.

Analysis

The state equations are

$$\dot{x} = f(x,y,u,t) \tag{1}$$

$$\epsilon\dot{y} = g(x,y,u,t) \tag{2}$$

where x, y, and u are scalars. It is desired to minimize the final value of x, $x(t_f) = x_f$, subject to $x(t_o) = x_o$, $y(t_o) = y_o$, $y(t_f) = y_f$, t_o and t_f fixed; this is the so-called "classical problem of Mayer" in two state variables. It is assumed that both the original problem, for which $\epsilon = 1$, and the reduced problem, where $\epsilon = 0$, have solutions. An approximation to the former is sought in terms of an expansion in powers of ϵ about the latter.

The Euler-Lagrange equations are given in terms of

$$H \equiv \lambda_x f + \lambda_y g \tag{3}$$

as

$$\dot{\lambda}_x = -\partial H/\partial x \tag{4}$$

$$\epsilon\dot{\lambda}_y = -\partial H/\partial y \tag{5}$$

$$\partial H/\partial u = 0 \tag{6}$$

and the transversality condition is $\lambda_{xf} = -1$. For $\epsilon = 0$, the variable y loses its exalted status; the order of the differential equations drops. One speaks of this as the "reduced system"[1] of equations. Its solution may be distinguished notationally by superscribed bars. It will be assumed, in order to avoid the complications of various singularity phenomena, that the strengthened form of the Legendre-Clebsch condition is satisfied for the reduced problem

$$(H_{yy}g_u^2 - 2H_{uy}g_u g_y + H_{uu}g_y^2) < 0 \tag{7}$$

This, together with smoothness assumptions on the functions f and g, insures $\bar{y}(t)$ and $\bar{u}(t)$ continuous—no "corners."

In problems well suited to approximate solution by asymptotic expansion, the reduced system's solution will strongly resemble that of the original except near the endpoints, where jumps in y occur with the reduced-order model to satisfy the end conditions. These abrupt transitions belie the assumption of small $\dot{y}$ which was invoked to obtain the reduction in order. It is consistent to replace the jumps by an approximation to y for t near t_o and t_f, which is also based upon an expansion in ϵ. A time variable

$$\tau = (t - t_o)/\epsilon \tag{8}$$

is introduced and the state-Euler system becomes, for $\epsilon = 0$

$$(d/d\tau)\lambda_x = 0 \tag{9}$$

$$(dx/d\tau) = 0 \tag{10}$$

$$(d/d\tau)\lambda_y = -\partial H/\partial y \tag{11}$$

$$(dy/d\tau) = g \tag{12}$$

$$(\partial H/\partial u) = 0 \tag{13}$$

This is the "boundary-layer system," whose solution may be denoted by $x^*(\tau)$, $y^*(\tau)$, $u^*(\tau)$. Evidently $x^*(0) = \bar{x}(t_o) = x_o$ and $\lambda_x^*(0) = \bar{\lambda}_x(t_o)$. It is noted that the boundary-layer equations are the state and Euler equations for the minimization of the integral performance index

$$-\int_0^\infty \lambda_{xo} f(\bar{x}_o,y^*,u^*,t_o)d\tau \tag{14}$$

subject to

$$(d/d\tau)y^* = g(\bar{x}_o,y^*,u^*,t_o)$$

and

$$y^*(0) = y_o,\ y^*(\infty) = \bar{y}(t_o) \tag{15}$$

The question of initial value for λ_y^* would seem hardly less trivial than for the original nonlinear two-point boundary problem, but, if the value could be determined, however indirectly, the possibility of using the existing singular perturbation theory for initial value problems would then arise. For large τ, both λ_y^* and y^* must approach the initial values $\bar{\lambda}_y$ and $\bar{y}$ of the reduced problem if the right members of Eqs. (11) and (12) are to approach the zero values required for equilibrium. Thus, attention is directed to the limiting behavior in the boundary layer as $\tau \to \infty$.

To this end, the boundary-layer equations may be linearized

$$y^* = \bar{y}_o + \delta y,\ \lambda_y^* = \bar{\lambda}_{yo} + \delta\lambda_y,\ u^* = \bar{u}_o + \delta u$$

$$(d/d\tau)\delta\lambda_y = -g_y\delta\lambda_y - H_{yy}\delta y - H_{yu}\delta u \tag{16}$$

$$(d/d\tau)\delta y = g_y\delta y + g_u\delta u \tag{17}$$

$$H_{uy}\delta y + H_{uu}\delta u + g_u\delta\lambda_y = 0 \tag{18}$$

Eliminating δu from Eqs. (16) and (17) gives two first-order equations. The various coefficients are independent of τ; they depend only upon t as a parameter; time "stands still" in the boundary layer. This allows easy examination of stability in terms of the characteristic roots of the quadratic

$$s^2 - (a + d)s + (ad - bc) = 0 \tag{19}$$

where

$$a = -g_y + (H_{yu}/H_{uu})g_u \tag{20}$$

$$b = -H_{yy} + (H_{yu}^2/H_{uu}) \tag{21}$$

$$c = -g_u^2/H_{uu} \tag{22}$$

$$d = g_y - g_u(H_{yu}/H_{uu}) \tag{23}$$

Noting that $a + d = 0$, one obtains

$$s = \pm(bc - ad)^{1/2} = \pm[(H_{yy}g_u^2 - 2H_{yu}g_y g_u + H_{uu}g_y^2)/H_{uu}]^{1/2} \tag{24}$$

In view of the assumed definiteness of expression (7) and with the additional assumption $H_{uu} < 0$, the bracketed expression in Eq. (24) is positive, and the two roots are nonzero, real, equal in magnitude, and opposite in sign. The bracketed expression becomes negative when the Legendre-Clebsch condition for the reduced problem is violated at the initial point.

Tihonov's theory of the initial value problem rests heavily upon an assumption of asymptotic stability for the boundary-layer equations. The reader is referred to Secs. 39 and 40 of Ref. 1 for the analysis and the other assumptions, which are less restrictive from an applications viewpoint. The instability of the Euler boundary-layer equations for large τ, evi-

Received January 21, 1970; revision received May 2, 1970. Research performed under Contract NAS 12-656 with NASA Electronics Research Center, Cambridge, Mass.

* Vice President.

denced by the positive member of the pair of real roots, represents a difficulty in adapting the theory. The conclusion implies that the unspecified initial condition $\lambda_y^*(0)$ must be chosen to suppress the unstable component of the solution, if this is possible, and this is closely related to the "matching" problem of matched asymptotic expansions.

If one attempts to proceed with expansion when the roots are imaginary, oscillations are encountered in the boundary layer which cannot be suppressed for large τ by choice of a single initial value, and the procedure fails. The borderline case of repeated zero roots encounters secular growth, corresponding also to failure, and hence it appears that the strengthened Legendre-Clebsch condition at the endpoints is essential for success.

In the terminal boundary layer, stability for reversed time is required by the Tihonov theory but, again, one or the other of the pair of real roots will obviate it, and suppression of instability by the choice of an end condition will again be needed. Zero or imaginary roots again signal failure.

After calculating the solution of the reduced system, and then the initial and terminal boundary layer solutions, each in turn, and combining them à la Vasil'eva (Theorem 40.1 of Ref. 1), one then has an approximation to the solution. Higher-order approximations are also offered by the Vasil'eva theory. The value of approximations of various order in flight mechanics applications, such as turn and climb performance of aircraft,[2] remains to be assessed.

Example

The following simple example is of interest. Take $f = \frac{1}{2}(Ay^2 + Bu^2)$, $g = u$, and end conditions $y(0) = y(t_f) = 1$, $x(0) = 0$. The solution of the reduced problem is $\bar{u} = \bar{y} = \bar{x} = \bar{\lambda}_y = 0$, $\bar{\lambda}_x = -1$. The solution of the initial boundary-layer equations takes the form

$$y^*(\tau) = Ce^{\gamma\tau} + De^{-\gamma\tau} \tag{25}$$

where $\gamma \equiv (A/B)^{1/2}$ and the constants C and D must be chosen to satisfy $y^*(0) = 1$, and $y^* \rightarrow \bar{y} = 0$ for large τ. This illustrates the selection of the undetermined multiplier initial value (buried in C and D) to suppress instability in the boundary layer, i.e., $C = 0$, $D = 1$. The situation in the terminal layer is similar, except that the "stable" component is suppressed. The composite solution, representing an expansion to zero-order in ϵ, is

$$\hat{y}(t) = e^{-\gamma\tau} + e^{\gamma(t-t_f)} \tag{26}$$

which may be compared with the exact solution

$$y = [(1 - e^{-\gamma t_f})e^{\gamma t} + (e^{\gamma t_f} - 1)e^{-\gamma t}]/(e^{\gamma t_f} - e^{-\gamma t_f}) \tag{27}$$

The approximation appears to be good for t_f large and/or γ large.

Concluding Remarks

The preceding analysis of a simplified two-state-variable case suggests that, when the Legendre-Clebsch condition for the reduced problem is met in strengthened form at the endpoints, initial values for boundary-layer equations should be chosen to suppress unstable solution components, if possible. Further work is needed to explore the possibility of suppression. The appearance of singular points and/or arcs in the interior of reduced system solutions is also of interest. Extensions of the analysis to more general problems, and particularly to those with more state variables, are obviously needed.

References

[1] Wasow, W., *Asymptotic Expansions for Ordinary Differential Equations*, Interscience (Wiley), New York, 1965.

[2] Kelley, H. J. and Edelbaum, T. N., "Energy Climbs, Energy Turns and Asymptotic Expansions," *Journal of Aircraft*, Vol. 7, No. 1, Jan.-Feb. 1970, pp. 93–95.

Linear Instability of Far-Wake Flow

Toshi Kubota* and Wilhelm Behrens†
California Institute of Technology, Pasadena, Calif.

Nomenclature

b = wake half-width defined as width from wake axis to point where $\Delta u = 0.5\Delta u_{℄}$ (physical coordinates)
$\bar{b}$ = wake half width in Howarth coordinates
c = wave velocity ($= c_R + ic_I$)
c_I = dimensional amplification factor
d = cylinder diameter
$\tilde{e}_f$ = rms hot wire fluctuation voltage at frequency f
M_e = wake edge Mach number
ΔM = relative Mach number $[= (u_e - u_{℄})/a_e]$
$\tilde{m}'$ = rms mass flow fluctuation
Re_d = freestream Reynolds number based on cylinder diameter
ΔT = temperature excess $[= (T_{℄} - T_e)/T_e]$
x = axial distance from cylinder
y = normal distance from wake axis
α = dimensional wave number

Subscripts

e = wake edge value
℄ = wake centerline value

Introduction

MEASUREMENTS of the mean wake flow behind circular cylinders at Mach 6, described in Ref. 1, showed that in a certain Reynolds number range ($Re_d \simeq 300$–4000) the inner wake stemming from the body boundary layers is laminar and loses its identity within 60 diam, and the outer wake, stemming from the bow shock, is also laminar within a certain downstream distance, and may be calculated from a laminar linear theory, but does not become similar (gaussian) within 2400 diam. Experimental study of the stability of these wakes is presented in Ref. 2.

In Ref. 2, the amplification rates of fluctuations measured in the regimes where the mean flowfield is predicted by laminar two-dimensional theory were compared with amplification rates calculated from the linear two-dimensional stability theory of compressible wakes by Lees and Gold.[3] In the theory, mean profiles of velocity and temperature were assumed to be gaussian distribution in the Howarth-transformed variable. The most unstable frequencies were predicted quite well but the measured rates of amplifications were somewhat lower than the theoretical values (see Fig. 10, Ref. 2). At the time, it was suggested that the discrepancy between the experimental results and stability theory was caused by the discrepancy in the mean wake profile shape.

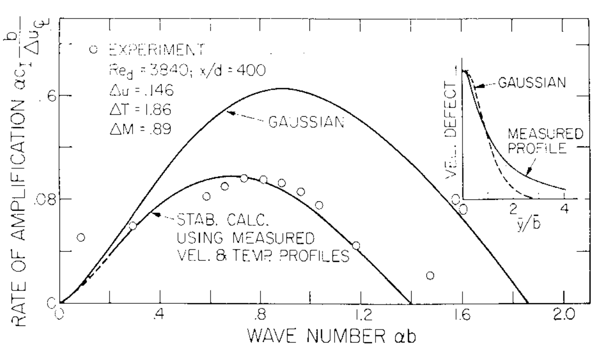

Fig. 1 Experimental and theoretical amplification rates at $Re_d = 3840$; $x/d = 400$.

Received March 5, 1970. This work was carried out under the sponsorship and with the financial support of the U.S. Army Research Office and the Advanced Research Projects Agency under Contract DA-31-124-ARO(D)-33, part of Project DEFENDER sponsored by the Advanced Research Projects Agency.

* Associate Professor of Aeronautics. Member AIAA.
† Assistant Professor of Aeronautics. Member AIAA.